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# ESTIMATING RELIABILITY AS A FUNCTION OF STRESS/STRENGTH DATA

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ARMY ROCKET AND GUIDED MISSILE AGENCY

U.S. ARMY ORDNANCE MISSILE COMMAND, REDSTONE ARSENAL, ALABAMA



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# ABSTRACT

The mode of failure for guided weapons is shown to be by chance excess of the failure strength of some component of the assemblage rather than by wear out. An expression for the probability of such failure is developed utilizing the assumption of the normal density function for the description of the scatter of maximum stress experience and of failure strength of a component. The effect of reduction of scatter is shown, both for a single component and for a multiplicity of components.

## Stress Failure Distribution

The failure of an element of equipment such as a structural member or a soldered joint due to overloading is in general of the nature of chance failure (Ref. 1), since (neglecting long-term effects such as fatigue) if the beam survives the stressing, it is just as good as new. This situation gives rise to a failure probability corresponding to the probability of disintegration of a radioactive nucleus. The element, like the nucleus, has no memory and after surviving to time  $t$  is just as good as it was at time 0. The radioactive decay equation, expressed in symbols familiar to the reliability specialist, describes the time behavior of failure of the element:

$$\phi_1(t) = \frac{-1}{N(0)} \frac{dN(t)}{dt} = \frac{1}{\bar{t}} e^{-t/\bar{t}} \quad (1)$$

where  $N(t)$  is the number of units that have not failed at time  $t$  and  $\bar{t}$  is the mean time to failure. The equation is in the normalized form and  $\phi_1(t)$  is expressed in fractional probability. The functional behavior of  $\phi_1(t)$  is shown in Figure 1.

If the element is used a comparatively large number of times so that it begins to wear out, failure of a different type occurs. Just before failure in this mode, the element is not as good as new. Its life expectancy is, on the average, quite short at this point and we may expect early failure. Failure in this mode will scatter about a mean in such a way as to approach the normal or Gaussian distribution, which expresses the distribution of random fluctuations in physical phenomena. The functional behavior of  $\phi_2(t)$  is shown in Figure 2 and is given by the equation:

$$\phi_2(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\bar{t})^2}{2\sigma^2}} \quad (2)$$

where  $\sigma$  is the dispersion expressed as standard deviation.

If the chance failure curve is combined with the wear-out curve, the function of Figure 3 is obtained. Here the shape of the curve is similar to the density function expression for human mortality, which is also dotted in Figure 3. The shapes of the two curves are similar; the major difference is the very high initial value of the human mortality function. This initial behavior is associated with accidents at birth, absence of immunity to disease at an early age, and with abnormal weaknesses of the very young, and has no counterpart in equipment being discussed here unless the design or manufacture of the equipment is faulty.

If we limit our discussion to short-lived weapons such as guided missiles, wear out failures are in general negligible. The exceptions are for parts subjected to severe environmental conditions. An example is the engine of the V-1, which had a nominal life of only 30 minutes because of severe high frequency vibration. Another is the case of jet vanes exposed to hot exhaust gases. Still another exception is the case of the absurd "testing to death" on the launching pad before firing.

We are then left with chance failure as the primary mode of failure of significance in the proper operation of the guided weapon. Is it possible to elicit additional information by analysis?

### Stress Levels

If we examine the loading of an element such as a beam we observe that there is some nominal maximum stress to which the beam is ordinarily subjected. Here, too, the element of chance enters, and a distribution about whatever maximum stress is selected will surely be observed. The normal law surely does not exactly fit the stress experience, but will give a satisfactory approximation at least for semi-quantitative discussion; it therefore will be assumed to suitably

describe the scatter about the nominal maximum. That the stress required to cause failure of the element will similarly vary about a nominal value is well known to those who have concerned themselves with quality control. In the case of those elements for which a failure strength is established as a result of physical testing, the maximum of the probability function will be located at a higher value than the nominal because strength is arbitrarily defined to be that of the weakest member of the group of elements tested. The argument here will nevertheless be valid with appropriate modification because this procedure does not alter the existence of the distribution of failure strength, but merely adds to the factor of safety.

Figure 4 demonstrates the result of the scatter for both maximum stress experience expectancy and failure strength expectancy. The nominal maximum stress experienced in a test is assumed to be 10 units; the stressed member is designed for failure at a nominal 20 units, giving a factor of safety of 2. We observe that for a load stress corresponding to  $f'$ , all elements characterized by a failure strength of  $f'$  or less will fail; for a failure strength greater than  $f'$ , the element will survive. Beyond this value, the stress falls short of causing a failure. Using the normal density function (2); the failure probability is represented by the expression:

$$Q(a, b) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left[\frac{-(f'-a)^2}{2\sigma_1^2}\right] df' \int_{-\infty}^{f'} \exp\left[\frac{-(f''-b)^2}{2\sigma_2^2}\right] df'' \quad (3)$$

where  $a$  is the stress corresponding to the nominal maximum stress experience;  $b$  the stress corresponding to the nominal failure strength; and  $\sigma_1$  and  $\sigma_2$  are the standard deviations of the stress experience and failure strength curves, respectively. The failure probability function  $Q(a, b)$  can be integrated by appropriate change of variable, with the result (3):

$$Q(a, b) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \int_{b-a}^{\infty} \exp\left[\frac{-f'^2}{2(\sigma_1^2 + \sigma_2^2)}\right] df' \quad (4)$$

This integral is easily evaluated using a table of the cumulative normal distribution. Figure 5 presents the graphical interpretation of  $Q(a, b)$ .

## Stress-Strain Variability Relationships

Examination of the failure probability function reveals that, once the nominal values  $a$  and  $b$  are fixed, it is a function of the standard deviations, these quantities fixing shape and therefore the extent of overlap of the curves. The significance of this statement necessitates repetition. Once the nominal values of maximum stress experience and of failure strength are fixed, the standard deviations  $\sigma_1$  and  $\sigma_2$  determine the value of the failure probability function  $Q(a, b)$ . Furthermore, the function is highly sensitive to the values of the standard deviations.

Figures 4 and 6 illustrate the effect of the magnitude of the standard deviations on the shape of the curves. Each of the curves is normalized; that is, the total area under the curves is unity when the abscissa is expressed in units of standard deviation. In the example represented by the curves of Figure 4 and nominal maximum value of stress experience is  $a = 10$  stress units; the nominal value of failure stress is  $b = 20$  stress units. The standard deviations  $\sigma_1$  and  $\sigma_2$  are each equal to 4 stress units. The value of the function  $Q(a, b)$  is 0.039, corresponding to a reliability of 96 per cent. In the example represented by the curves of Figure 6, nominal stress values are unchanged, but standard deviations are reduced to one half of the former value, e. g. 2 stress units. The curves are much more sharply peaked about the nominal values. Evaluation of the failure probability function for this case yields a failure rate of 2 per 10,000, down by a factor of nearly 200 from that of the first example. Further reduction of each of the standard deviations to one stress unit vastly improves the reliability, the failure rate being 1 in 1 trillion. If the density functions were compressed to the limit, i. e., a delta function about the nominal values, the failure probability function would clearly approach zero as a limit. The low value obtained for  $Q(a, b)$  in the third example demonstrates that the failure expectancy becomes vanishingly small long before delta function limit is reached.

The purpose of this discussion is to elucidate the effect of scatter on the failure probability function. No one will be so foolish as to purposely design a part with a failure stress less than the expected stress. The failure is a result of scatter. Control of this scatter, or more specifically, the magnitude of the standard deviations is the key to the control of reliability of the unit under consideration. The magnitude of the maximum stress experience standard deviation may be, at



least in theory, completely at our disposal, such as in the case of screaming in a rocket engine; it may be completely independent of our will, as in the case of countermeasures against our weapon. In the first case, as our engineering of the combustion process becomes increasingly precise, we reduce the standard deviation of the maximum stress experience curve by being able to more closely specify just what the maximum chamber pressure will be. In the second case, about all that can be done is to specify, for example, the overpressure at which we will permit an airframe structure to fail.

The magnitude of the failure strength standard deviation is completely at our disposal, within the limits of our design and production knowledge. The control of this scatter is the proper function of quality control activity. It is, however, a matter of primary importance that the quality control apply to the actual conditions of use of the item. For example, the scatter of failure stress for a non-screaming rocket engine would be expected to be less than that of one that screams because of the greater heat transfer coefficient and the concomitant higher wall temperature of the latter.

### Example

To summarize the discussion of the scatter of the maximum stress experience and failure strength functions, we may state that good engineering judgment and practice, including precise quality control, are the effective means to control of reliability. When the functions can be closely specified, the factor of safety can be reduced to a low value and reliability maintained. Conversely, when scatter of these functions is large, the price of our ignorance is an increased factor of safety or a reduced reliability, or some combination of the two.

At this point a mention of the customarily discussed compound probability relation is in order. The over-all reliability of an assembly of units which are assumed to be mutually independent and which must all function to avoid failure is the product of the reliabilities of the individual units. Thus, the over-all reliability of 10 of the units of 99.98 per cent individual reliability of example 2 above is about 99.8 per cent and still better than the single unit of example 1. Over-all reliability of 100 units is 98 per cent, and of 1000, 82 per cent. For the third example, the compound probability of failure for 1000 units

is only about 1 in 1 billion. The importance of scatter control is seen to be overwhelming; by reducing the standard deviations by a factor of four, failure expectancy is reduced by a factor of several million. The other obvious conclusion is that reduction of the number of elements of an assembly is of exponential weight in the reliability problem. Redundancy of critical components is an extreme measure that can be used to reduce the failure expectancy. It is effectively applied in the case of multi-engine aircraft, however.

### Summary

To summarize, we have observed that the failure of components of guided weapons is generally in the nature of chance rather than wear-out failure. A failure probability function was developed to show the interrelation of maximum stress encountered, component failure strength, and the scatter of the values of these parameters. It was shown that the failure probability function was highly sensitive to scatter, and that where a multiplicity of components was considered, a moderate reduction of scatter could result in the difference between success and failure. The same result can be achieved by reduction of complexity.

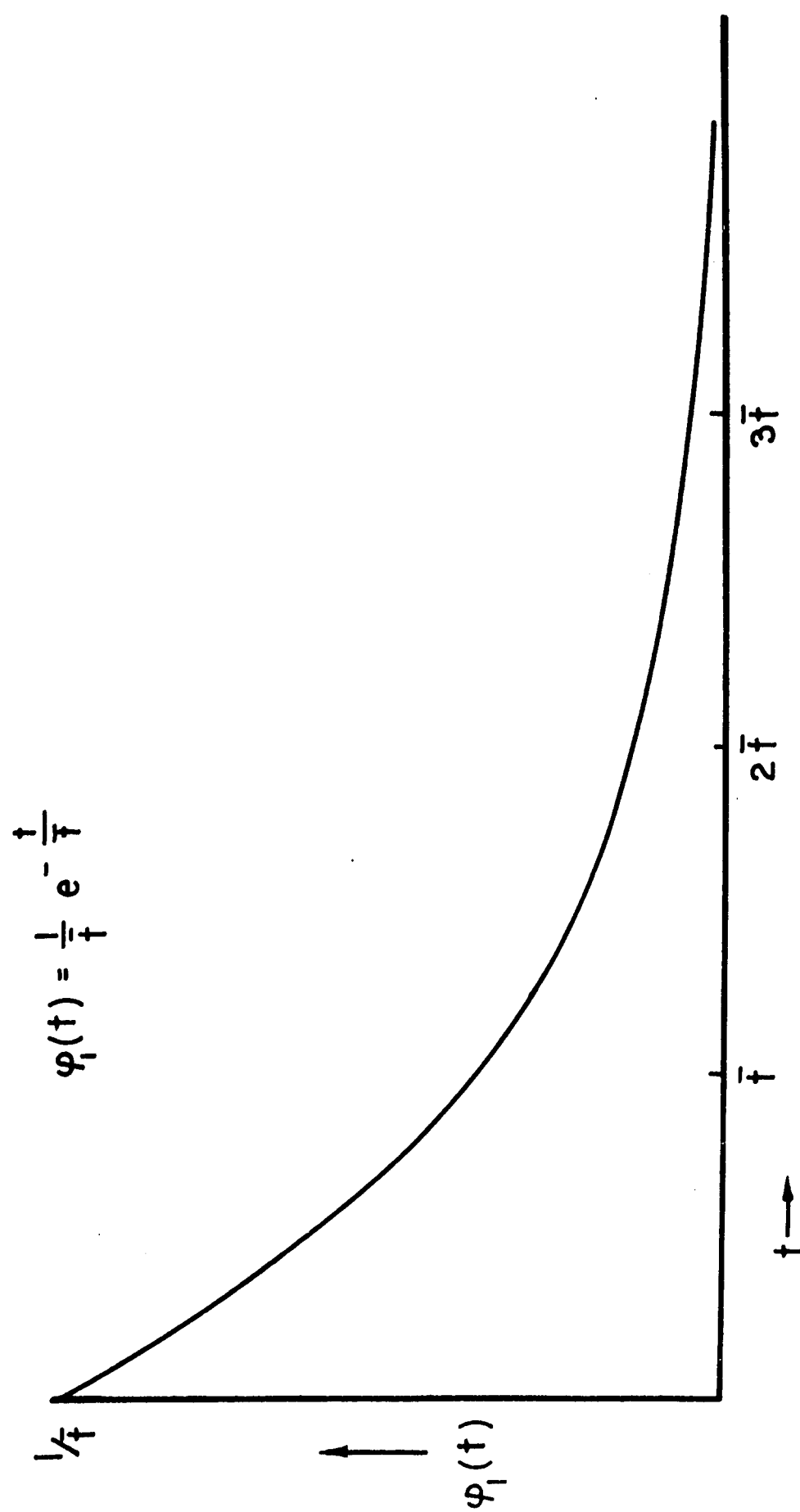


FIG. 1.- EXPONENTIAL CHANCE FAILURE LAW

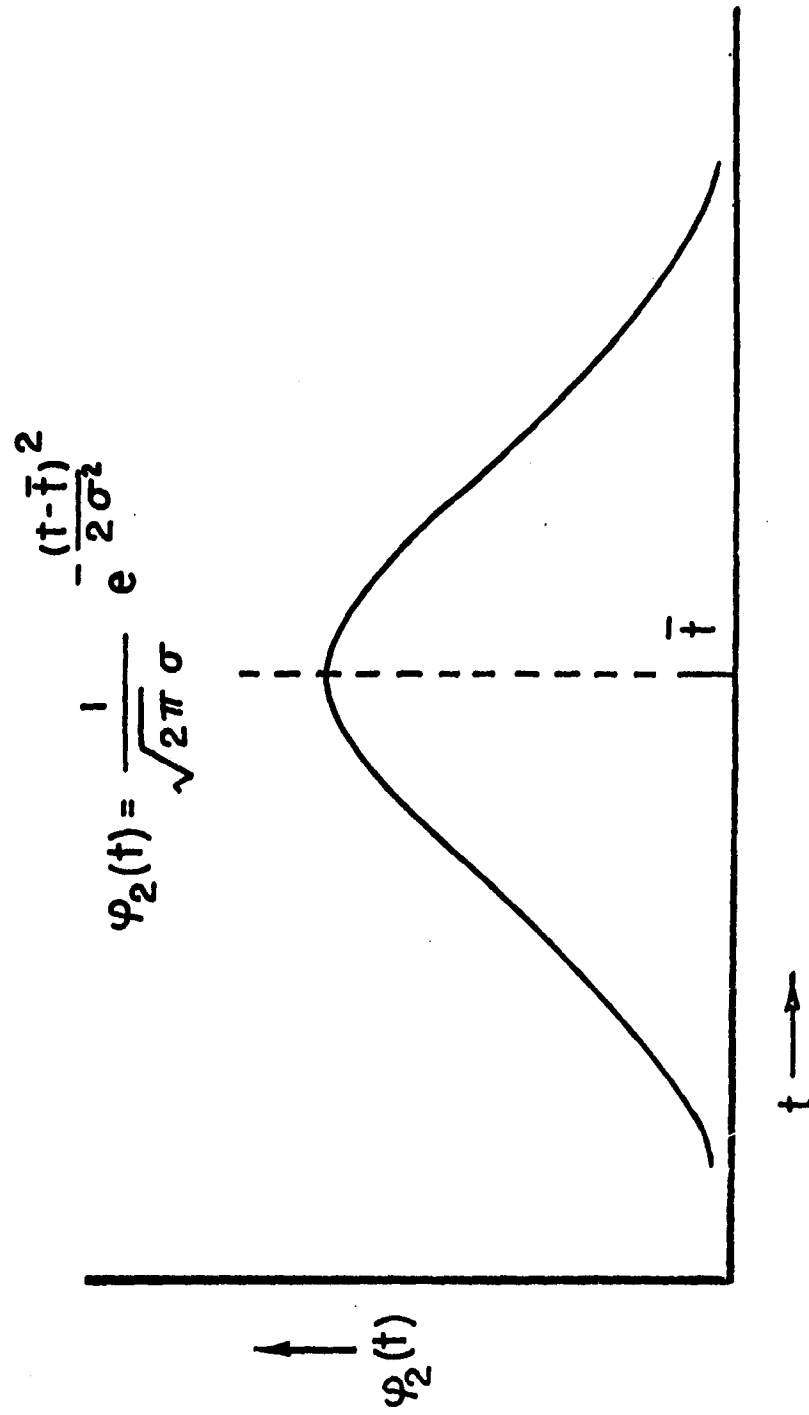


FIG. 2.- WEAR OUT FAILURE LAW

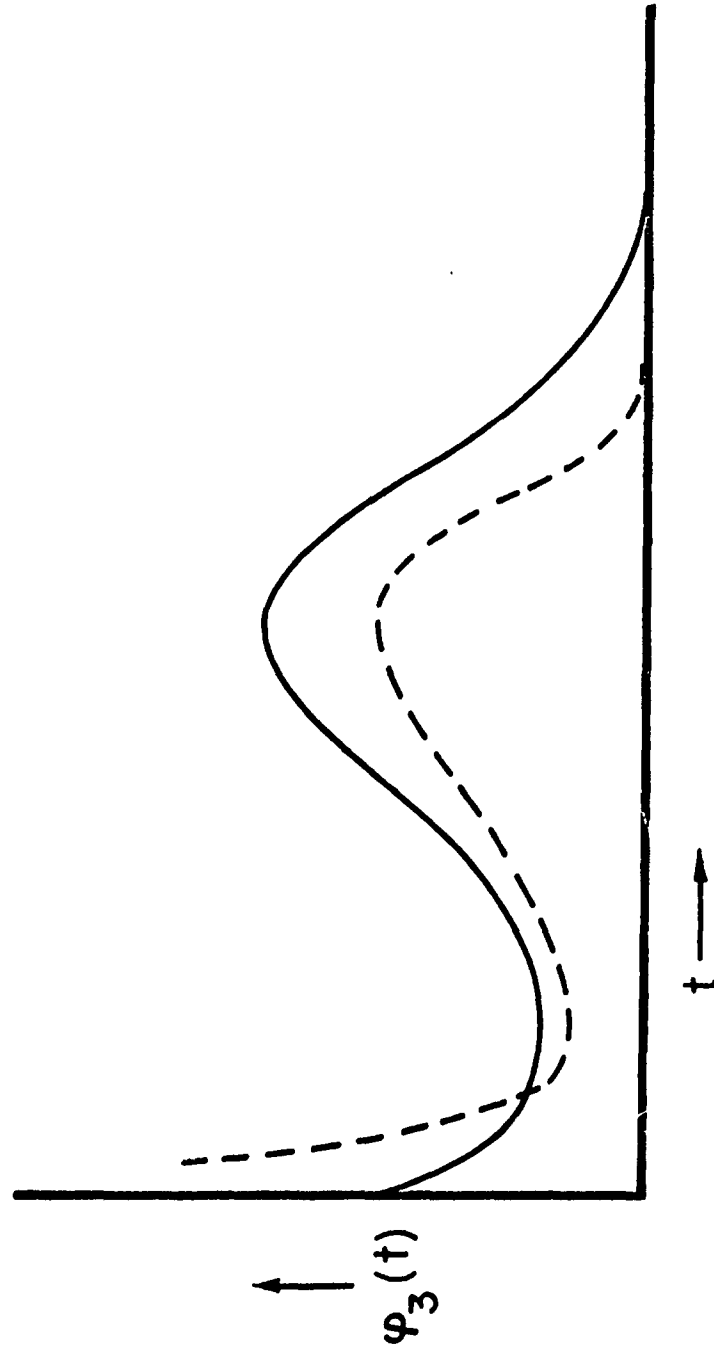


FIG. 3.- COMPOUND FAILURE LAWS

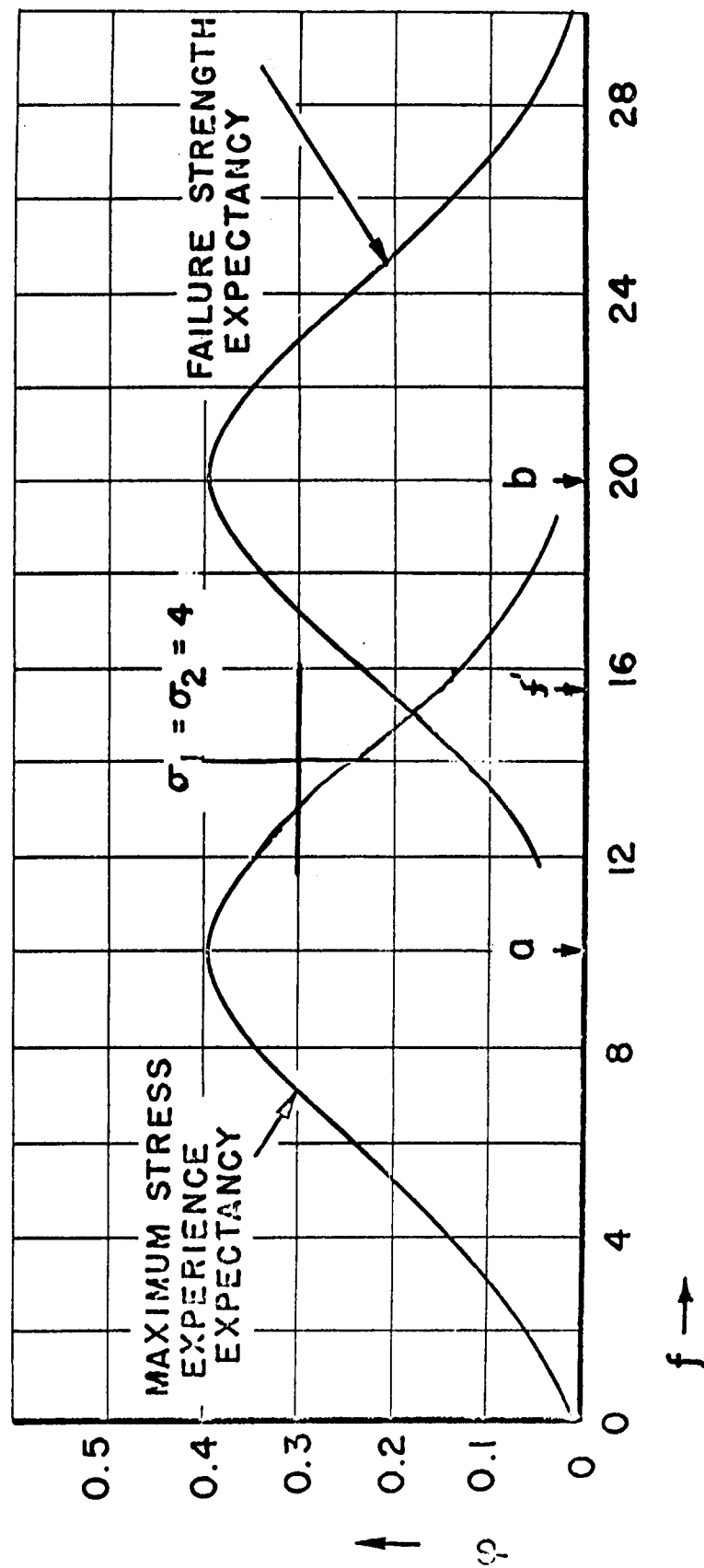


FIG. 4.-NORMAL DENSITY FUNCTIONS FOR MAXIMUM STRESS EXPERIENCE AND FAILURE STRENGTH

$$Q(a,b) = \frac{1}{\sqrt{2\pi}(\sigma_1^2 + \sigma_2^2)} \int_{b-a}^{\infty} \exp \left[ \frac{-f'^2}{2(\sigma_1^2 + \sigma_2^2)} \right] df'$$

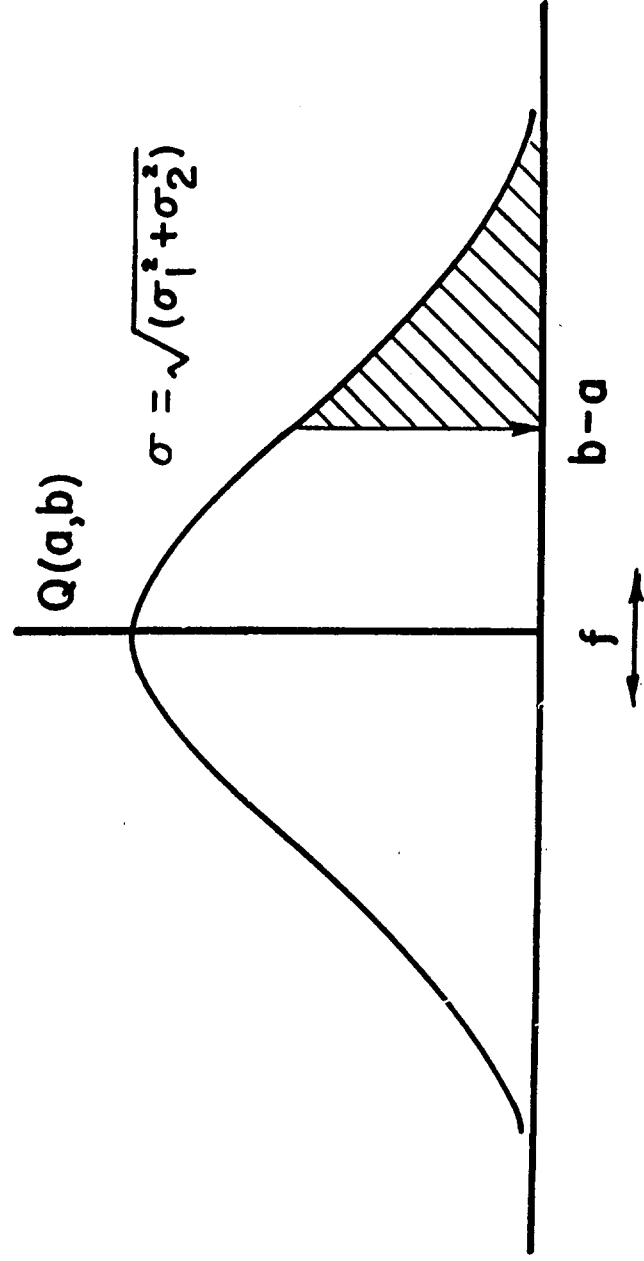


FIG. 5.- FAILURE PROBABILITY FUNCTION  $Q(a,b)$

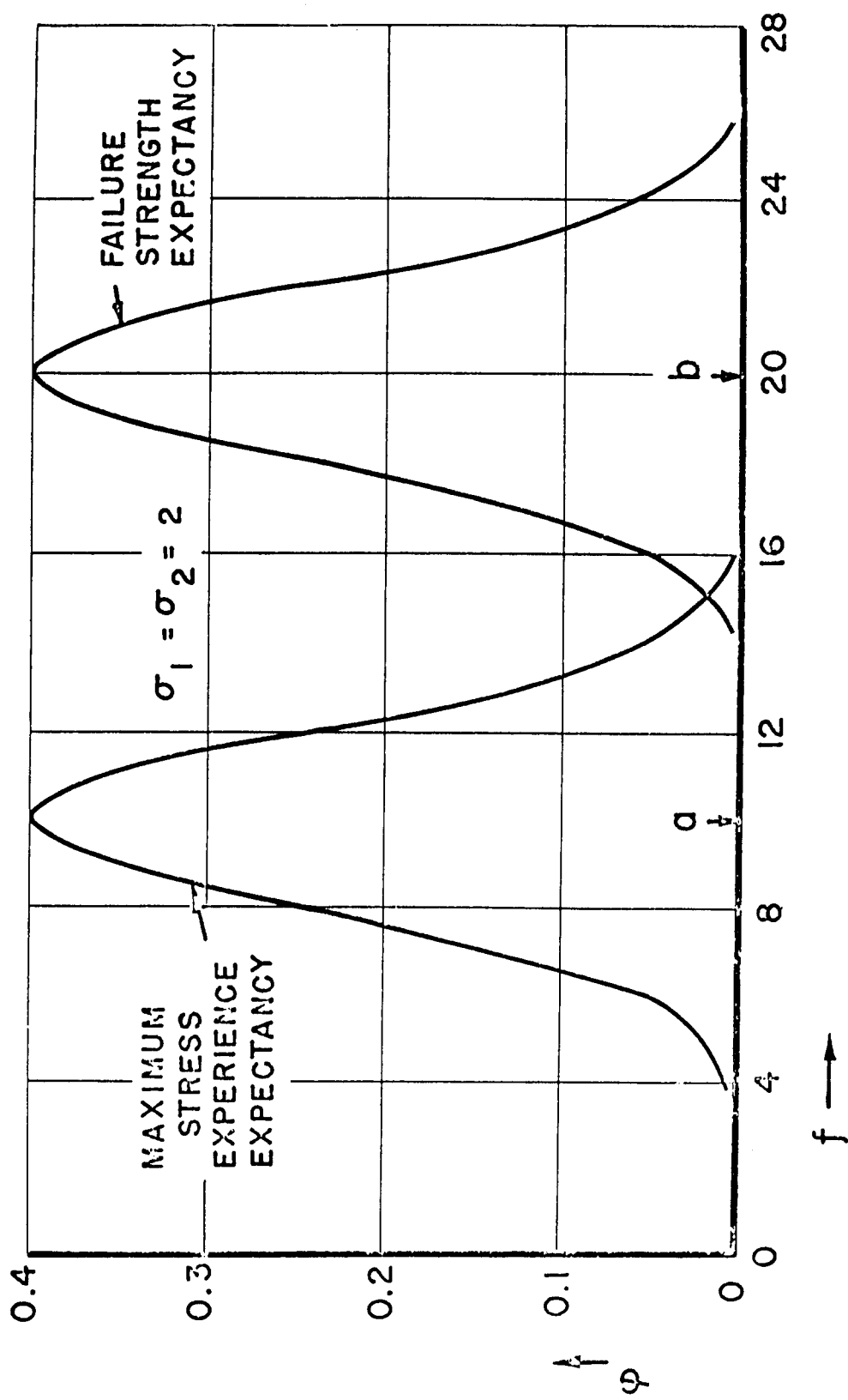


FIG. 6.- EFFECT OF REDUCED SCATTER ON  
NORMAL DENSITY FUNCTIONS



## Appendix A

By E. G. Pieruschka

### Classical Theory of a Combination of Univariate Stress and Strength Distributions

A combination of univariate stress-strength distributions is shown in figure 7.

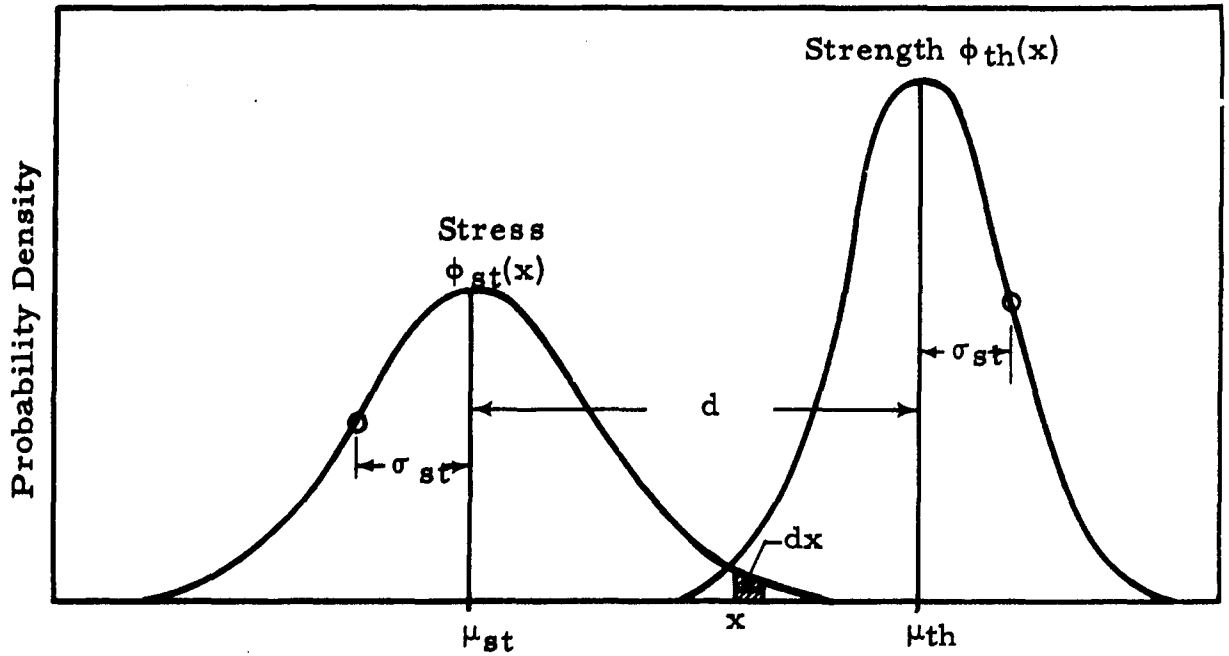


Figure 7. Stress-Strength Distribution.

Here we have a stress distribution  $\phi_{st}(x)$  with the mean  $\mu_{st}$ . The strength of the device type varies with device to device according to the strength distribution  $\phi_{th}(x)$  with the mean  $\mu_{th}$ . A given device may have the strength  $x_{th}$ . The stress during the trial of the device may be  $x_{st}$ . When  $x_{th}$  is larger than  $x_{st}$  we have a successful trial. When  $x_{th}$  is smaller than  $x_{st}$  we have a malfunction. Thus, the probability of malfunction  $Q$  may be obtained by computing

$$Q = \int_{x=-\infty}^{\infty} \int_{z=-\infty}^x \phi_{th}(z) \phi_{st}(x) dz dx \quad (1.1)$$

The integration area of the two dimension function  $\phi_{st}(x) \phi_{th}(z)$  is indicated on figure 8:

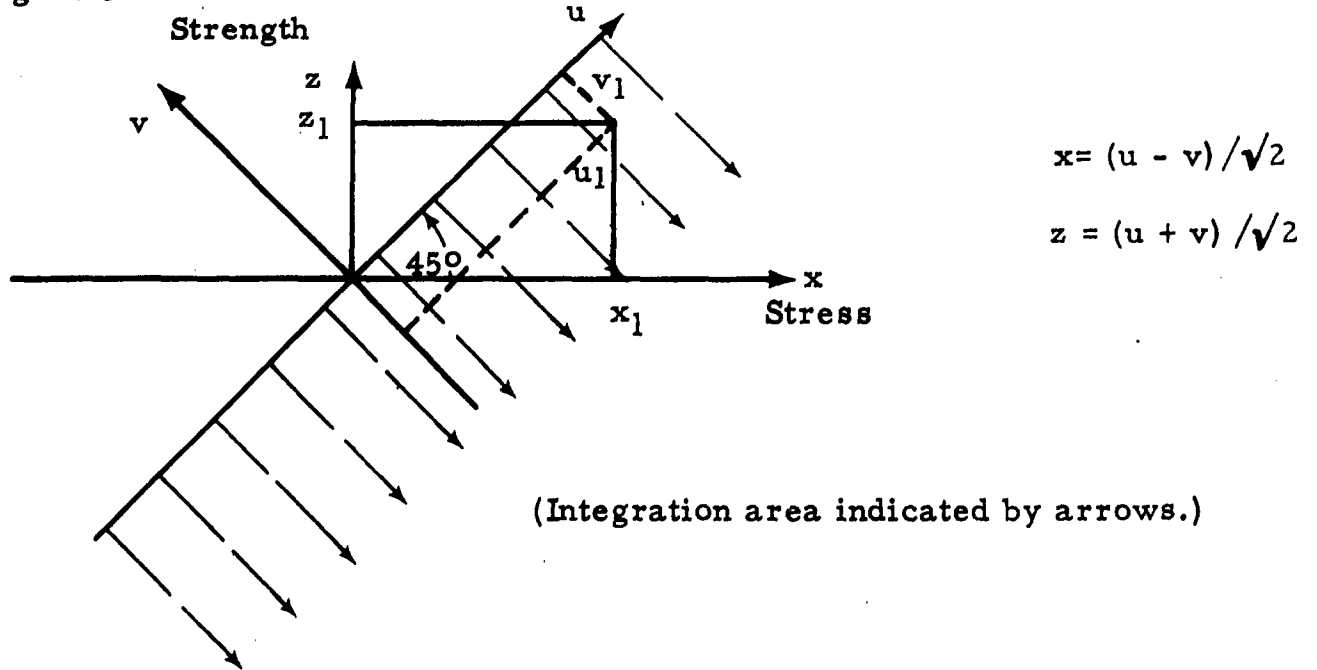


Figure 8 - The Coordinate Transform.

We recognize from figure 8 that the double integral (1.1) which has one variable integral limit may be transformed in a double integral with fixed integral limits:

$$Q = \int_{v=-\infty}^0 \int_{u=-\infty}^{\infty} \phi_{st}\left(\frac{u-v}{\sqrt{2}}\right) \phi_{th}\left(\frac{u+v}{\sqrt{2}}\right) du dv \quad (1.2)$$

When  $\phi_{st}(x)$  and  $\phi_{th}(z)$  are normal distributions we may write them as:

$$\phi_{st}(x) = \frac{1}{\sqrt{2\pi} \sigma_{st}} \exp \left[ -\frac{1}{2\sigma_{st}^2} (x - \mu_{st})^2 \right] \quad (1.3)$$

$$\phi_{th}(z) = \frac{1}{\sqrt{2\pi} \sigma_{th}} \exp \left[ -\frac{1}{2\sigma_{th}^2} (z - \mu_{th})^2 \right] \quad (1.4)$$

Putting (1.3) and (1.4) into (1.2) we have

$$Q = \frac{1}{2\pi\sigma_{st}\sigma_{th}} \int_{v=-\infty}^0 \int_{u=-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \left[ \left( \frac{u-v}{\sqrt{2}\sigma_{st}} - \frac{\mu_{st}}{\sigma_{st}} \right)^2 + \left( \frac{u+v}{\sqrt{2}\sigma_{th}} - \frac{\mu_{th}}{\sigma_{th}} \right)^2 \right] \right\} du dv \quad (1.5)$$

The quantities in parentheses may be written as

$$\left( \frac{u-v - \sqrt{2} \mu_{st}}{\sqrt{2} \sigma_{st}} \right) \text{ and } \left( \frac{u+v - \sqrt{2} \mu_{th}}{\sqrt{2} \sigma_{th}} \right), \text{ respectively.}$$

In order to simplify the left-hand parenthesis, we substitute

$$u = \sqrt{2} \sigma_{st} y + v + \sqrt{2} \mu_{st}$$

$$du = \sqrt{2} \sigma_{st} dy$$

Equation (1.5) becomes

$$Q = \frac{1}{\sqrt{2} \Pi \sigma_{th}} \int_{v=-\infty}^0 \int_{y=-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \left[ y^2 + \left( y \frac{\sigma_{st}}{\sigma_{th}} + \frac{2v - \sqrt{2} (\mu_{th} - \mu_{st})}{\sqrt{2} \sigma_{th}} \right)^2 \right] \right\} dy dv \quad (1.6)$$

$$\text{Let } \mu_{th} - \mu_{st} = d, \quad (1.7)$$

$$\sigma_{th}^2 + \sigma_{st}^2 = \sigma^2 \quad (1.8)$$

$$Q = \frac{1}{\sqrt{2} \Pi \sigma_{th}} \int_{v=-\infty}^0 \int_{y=-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \left[ y^2 \frac{\sigma^2}{\sigma_{th}^2} + 2y \frac{\sigma_{st}}{\sigma_{th}^2} \left( \frac{2v - \sqrt{2} d}{\sqrt{2}} \right) + \left( \frac{2v - \sqrt{2} d}{\sqrt{2} \sigma_{th}} \right)^2 \right] \right\} dy dv \quad (1.9)$$

On simplifying, the exponent becomes

$$\begin{aligned} & -\frac{1}{2} \frac{\sigma^2}{\sigma_{th}^2} \left[ y^2 + 2y \frac{\sigma_{st}}{\sigma^2} \left( \frac{2v - \sqrt{2} d}{\sqrt{2}} \right) + \frac{1}{2\sigma^2} (2v - \sqrt{2} d)^2 \right] \\ & = -\frac{1}{2} \frac{\sigma^2}{\sigma_{th}^2} \left[ y + \frac{\sigma_{th}}{\sigma^2} \frac{2v - \sqrt{2} d}{\sqrt{2}} \right]^2 + \frac{1}{2} \frac{\sigma^2}{\sigma_{th}^2} \frac{\sigma_{st}^2}{\sigma^4} \frac{(2v - \sqrt{2} d)^2}{2} \\ & \quad - \frac{1}{2} \frac{\sigma^2}{\sigma_{th}^2} \frac{(2v - \sqrt{2} d)^2}{2\sigma^2} \end{aligned}$$

Equation (1.9) becomes

$$Q = \frac{1}{\sqrt{2} \pi \sigma_{th}} \int_{v=-\infty}^0 \int_{y=-\infty}^{\infty} \exp \left\{ - \frac{\sigma^2}{2\sigma_{th}^2} \left[ y + \frac{\sigma_{th}}{\sigma^2} \frac{2v - \sqrt{2} d}{\sqrt{2}} \right]^2 \right\} dy \exp \left\{ - \frac{1}{4\sigma^2} (2v - \sqrt{2} d)^2 \right\} dv \quad (1.10)$$

Since for any value of  $v$ ,

$$\int_{-\infty}^{\infty} \exp \left\{ - \frac{\sigma^2}{2\sigma_{th}^2} \left[ y + \frac{\sigma_{th}}{\sigma^2} \frac{2v - \sqrt{2} d}{\sqrt{2}} \right]^2 \right\} dy = \sqrt{2} \pi \sigma_{th} / \sigma,$$

Equation (1.10) becomes

$$Q = \frac{1}{\sqrt{\pi} \sigma} \int_{-\infty}^0 \exp \left\{ - \frac{(v - d/\sqrt{2})^2}{\sigma^2} \right\} dv \quad (1.11)$$

Let

$$\sqrt{2} v - d = t \quad dv = \frac{1}{\sqrt{2}} dt$$

$$Q = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{-d} e^{-\frac{t^2}{2\sigma^2}} dt \quad \text{where } \sigma = \sqrt{\sigma_{th}^2 + \sigma_{st}^2} \quad (1.12)$$

Due to symmetry of the normal distribution, the integration limits may be reversed with opposite sign, giving the same result as equation (1.3).

## Appendix B\*

By H. A. David  
Professor of Statistics  
Virginia Polytechnic Institute

### Probabilistic Derivation of Probability that Stress Exceeds Strength

Suppose that  $x$ , denoting stress, is normally distributed with mean  $a$  and standard deviation  $\sigma_1$ , and that  $y$ , denoting strength, is normally distributed with mean  $b$  and standard deviation  $\sigma_2$ . The probability of failure when stress exceeds strength is:

$$Q = \Pr (x > y) \quad (2.1)$$

$$= \Pr (x - y - a + b > -a + b) \quad (2.2)$$

$$= \Pr \left[ \frac{x - y - a + b}{\sqrt{(\sigma_1^2 + \sigma_2^2)}} > \frac{-a + b}{\sqrt{(\sigma_1^2 + \sigma_2^2)}} \right] \quad (2.3)$$

$$= \Pr \left( u > \frac{b - a}{\sqrt{(\sigma_1^2 + \sigma_2^2)}} \right) , \quad (2.4)$$

where  $u$  is unit normal.

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\*This appendix gives the same result as that of Appendix A, but through a different approach.

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3. National Bureau of Standards Report 1237, "Evaluation of Complex Systems in Guided Missiles", Corona Calif., Nov. 1953, p. 5.
4. Pieruschka, Erich, private communication